

Equivalence of rational links and 2-bridge links revisited

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ABSTRACT. In this paper we give a simple proof of the equivalence between the rational link associated to the continued fraction $[a_1, a_2, \dots, a_m]$, $a_i \in \mathbb{N}$, and the two bridge link of type p/q , where p/q is the rational given by $[a_1, a_2, \dots, a_m]$. The known proof of this equivalence relies on the two fold cover of a link and the classification of the lens spaces. Our proof is elementary and combinatorial and follows the naive approach of finding a set of movements to transform the rational link given by $[a_1, a_2, \dots, a_m]$ into the two bridge link of type p/q .

1. Introduction

The equivalence between the two bridge link of type p/q , introduced and classified by Schubert [10], and the rational links associated to a continued fraction $[a_1, a_2, \dots, a_m]$, introduced by Conway [3], is one example of the beautiful relations between knot theory and other mathematical subjects, in this case number theory. In an elementary course of knot theory, this relation captures the students attention and imagination, but the known proof requires advanced techniques from 3-manifold theory that are out of reach at that level. For this reason, we seek an elementary proof, that follows the naive approach of finding an algorithm to change one of the diagrams into the other.

In this paper we will transform the Conway diagram C of a rational link, associated to a continued fraction $[a_1, a_2, \dots, a_m]$, with $a_i \in \mathbb{N}^+$, into a diagram S of the two bridge link of type p/q , where p/q is the rational given by the continued fraction $[a_1, a_2, \dots, a_m]$. In this way we give an elementary proof of the equivalence between the rational link $[a_1, a_2, \dots, a_m]$ and the two bridge link of type p/q . This equivalence can be found in [1], but this proof requires to consider the two fold cover of the link and the classification of the lens spaces. Our proof differs completely of this approach, and instead, use the direct method of finding a set of moves, that can be described in a recursive way as a sequence of steps. In each step of the transformation process we are able to describe precisely the changes in the diagram and to produce a sequence of integers that keep track of the changes and will allow us to confirm that at the end of the transformation we obtain the diagram of the 2-bridge link of the right type. The result of this paper together with the results

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in [7] and [6], the complete classification of rational links and two bridge links is completed, without requiring advanced techniques of three dimensional topology.

The role of knot theory as a didactic tool, not only in undergraduate courses but also as a subject to develop mathematical awareness in high school students, requires some effort to present part of the theory in an appropriate level. This elementary proof of the equivalence of the two families of links follows this approach.

The technique used in the transformation of a rational diagram into a 2-bridge diagram can be implemented in the transformation of a link diagram given by a 6-plat into a 3-bridge diagram given by the Schubert presentation as a triple $(p/n, q/m, s/l)$, where p, n, q, m, s, l are integers, see [5]. This is a subject of current research.

In section 1 we describe the diagrams of rational links and 2-bridge links and fix some notation. In section 2 we introduce the transformation process and establish the main result. Section 3 has a collection of technical results, of combinatorial and computational nature, that will be used in Section 4 to prove the main result.

2. Diagrams and notation

Let us consider the continued fraction $[a_1, a_2, \dots, a_m]$, with a_i a positive integer, $1 \leq i \leq m$. Figure 1 shows the diagrams $C = C[a_1, a_2, \dots, a_m]$ of rational links associated to a continued fraction $[a_1, a_2, \dots, a_m]$, where the tangles in even positions are positive and the ones in odd positions are negative, see Fig. 1 a.

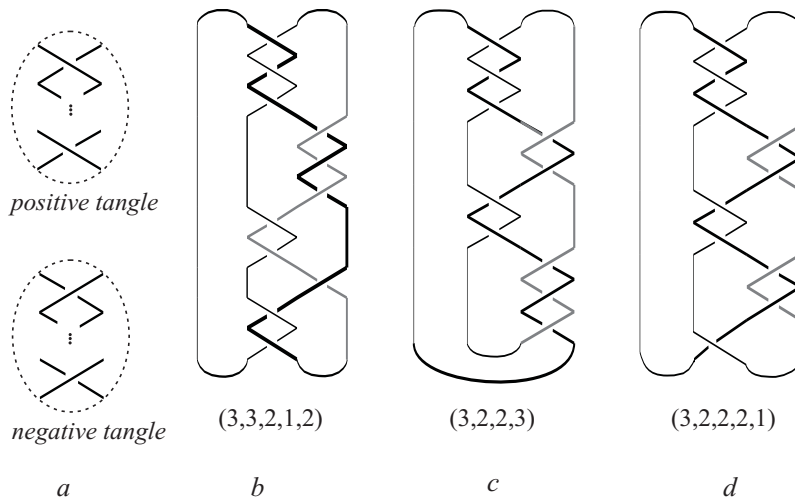


FIGURE 1. Rational links diagrams, with odd and even number of tangles.

The final arcs that close the diagram depend on m . If m is odd, we use the form shown in Fig. 1 b., and if it is even we use the form shown in Fig. 1 c. Usually we always can consider m odd, by a simple deformation of the diagram, as shown in Fig. 1 d., see [7], but in our work we need to consider both diagram types. Note that our convention for rational links follows [9], [4] and [7] and differs from the standard one given in [1], [3], [8] and [11], so our rational link $C[a_1, a_2, \dots, a_m]$ corresponds to the mirror image of the one in [1].

To a rational number p/q , with p and q relatively primes and $0 < q < p$, we associate a two bridge diagram $S = S(p/q)$ as shown in Fig. 2. We call α the bridge to the left and β the bridge to the right. The integer p represents the number of segments in which we divided each one of the bridges, so the number of crossings under each bridge is $p-1$. The integer q represents the position of the first crossing of the bridge α under the bridge β , as shown in Fig. 2 a., counting in a clockwise direction. See [8] and [10] for a more detailed description of the 2-bridge diagram.

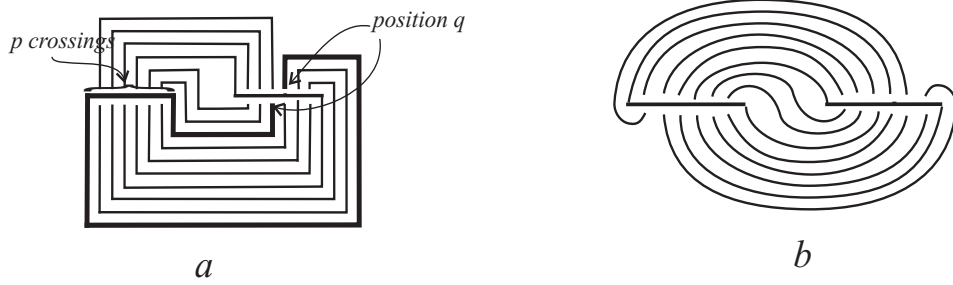


FIGURE 2. Two bridge link diagrams p/q . Particular case with $p = 7$ and $q = 3$.

Our 2-bridge link corresponds to the mirror image of the 2-bridge link in [10] and [1]. Usually the standard diagram of a 2-bridge link is symmetric, as shown in Fig. 2 b., but for our purpose we will consider the bridge β as formed by two segments, so the diagram in Fig. 2 a. is more appropriate.

We will describe a process to transform the diagram $C[a_1, a_2, \dots, a_m]$ into the diagram $S(p/q)$, when p/q is the rational given by the continued fraction $[a_1, a_2, \dots, a_m]$. The process will be defined in a recursive way. In each step n , with $1 \leq n \leq m$, where m is the length of the continued fraction, we construct a sequence of integers p_n and q_n and we prove that they satisfy the recurrence relation of the convergents of the continued fraction $[a_1, a_2, \dots, a_m]$, see [2],

$$(2.1) \quad p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n \geq 1,$$

with $p_0 = 1$, $p_1 = a_1$, $q_0 = 0$, $q_1 = 1$.

Notation: In all the paper, each a_i will be a positive integer. For a real number a , $\lceil a \rceil$, the ceiling of a , will be the least integer greater or equal to a , and $\lfloor a \rfloor$, the floor of a , will be the greatest integer less or equal to a . For an integer a we define

$$(2.2) \quad \mu_a = \begin{cases} 0 & \text{if } a \text{ is even} \\ 1 & \text{if } a \text{ is odd.} \end{cases}$$

We will use also the Kronecker delta δ_{lt} to indicate 1 if $l = t$ and 0 otherwise.

3. Geometric description of the transformation process

To transform the diagram C of a rational link into a two bridge diagram S , we take the two bridges as the two superior arcs of diagram C . We call α the bridge to the left and β the one to the right. From the two bridges emerge 4 strings, that wave to form the tangles that conforms the rational link. The string to the right does not play any role in the waving, so we will consider only 3 strings and we take

bridge α as formed by string 1 and divide bridge β into strings 2 and 3, so we will consider β as formed by two independent strings, see Fig. 3 a.

The transformation process will be a sequence of steps, one for each one of the tangles that form the rational link. In each step we transform the tangle so that all the crossings will be under the bridges. In each step of our process each string will play a different role. Two of them, that we will call the *guides*, are the ones that form the tangle and will be modified. The third one will be idle.

DEFINITION 1. We define as *principal guide* in step n the string that crosses over the other one in the first crossing of the tangle a_n , and we define as *secondary guide* the string that crosses under. See Fig. 4 a. The principal and secondary guides in step n are denoted by i_n and j_n respectively. The idle string is the one that does not form part of the tangle a_n and is denoted k_n .

To simplify notation we will drop the subscript and use only i, j and k when there is no place to confusion.

In order to count the number of crossings under each bridge we will keep track of the strings that form the crossings and we will count the number of times that string r crosses under string t , $1 \leq r, t \leq 3$. This information is stored in a data matrix $D_n = [d_{rt}(n)]_{r=1,3}^{t=1,3}$, where $d_{rt}(n)$ represents the times that string r crosses under string t . In each step of our process we transform the data matrix D_n into the matrix D_{n+1} .

Let us take the rational diagram C described by the continued fraction $[a_1, \dots, a_m]$. We start the process by taking

$$(3.1) \quad D_0 = [0]$$

and describe a recurrence process in order to change C into S .

3.1. Step One. In the first step we will unravel the first tangle a_1 as shown in Fig. 3 a. The tangle is formed by the 1 and 2 strings, with 1 as the principal guide, 2 as the secondary guide and 3 as the idle string. The movement changes the tangle with a_1 positive crossings into a 2-tangle in which there are only crossings under the bridges. The movement produces a_1 crossings under string 1, $\lfloor a_1/2 \rfloor$ of them formed by undercrossings of string 1 and $\lceil a_1/2 \rceil$ corresponding to crossings of string 2. Under string 2 we will have $a_1 - 1$ crossings, $\lceil (a_1 - 1)/2 \rceil$ of them formed by string 1 and $\lfloor (a_1 - 1)/2 \rfloor$ by string 2.

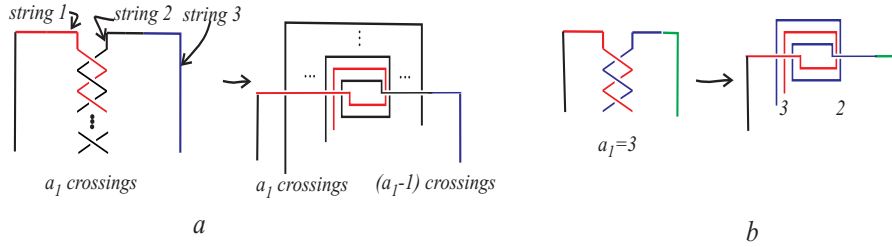


FIGURE 3. Step one, only strings 1 and 2 are used. Fig. b shows the case $a_1 = 3$.

Therefore, we have the following matrix to describe step one

$$D_1 = [d_{rt}(1)]_{r=1,3}^{t=1,3} = \begin{bmatrix} \lfloor \frac{a_1}{2} \rfloor & \lfloor \frac{a_1-1}{2} \rfloor & 0 \\ \lceil \frac{a_1}{2} \rceil & \lceil \frac{a_1-1}{2} \rceil & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The total number of crossings under string t is given by the sum $d_{1t} + d_{2t} + d_{3t}$, so these are the numbers that allow us to describe the 2-bridge link produced at the end of the process.

DEFINITION 2. For $t \in \{1, 2, 3\}$ we define $s_{n,t}$ as the sum of the t column of matrix D_n , i.e., $s_{n,t} = d_{1t} + d_{2t} + d_{3t}$.

3.2. Step $n+1$. Let us suppose that we have changed the tangles a_1, \dots, a_n and the information is stored in D_n .

In step $n+1$ we change the tangle a_{n+1} into a rational tangle, in the same way as we proceeded in step one. We have a_{n+1} crossings under the principal guide and $a_{n+1} - 1$ under the secondary guide, see Fig. 4 b. We move all the crossings, forming a parallel set of strings, following the direction of each of the guides, ending the movement when we reach the initial points of the guides. In this way all the crossings are under the bridges, see Fig. 4. Figure 6 shows the transformation process of $C[1, 2, 2]$. In Fig. 6b. we made the first step and transform the tangle 1. In Fig. 6c. we change the tangle 2 and in Fig. 6d. we move all the crossing, following the guides.

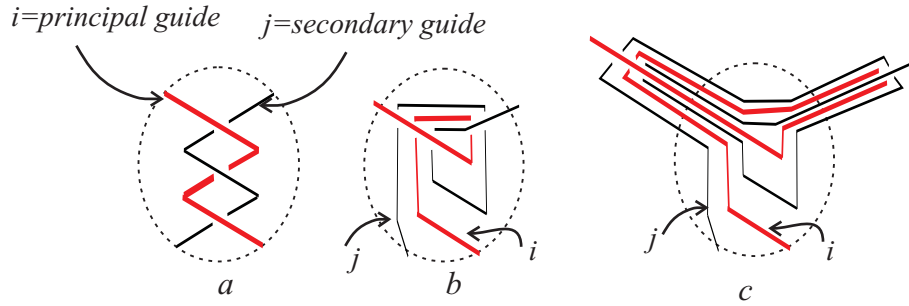


FIGURE 4. Chages in step $n+1$.

Now, the important part is to keep track of the number of crossings we have at the end of step $n+1$ and to find the new guides.

To simplify notation, take $a = a_{n+1}$ and let $D = D_n = [d_{rt}]$ and $D' = D_{n+1} = [d'_{rt}]$ defined below. Let us find the values of d'_{rt} in terms of d_{rt} and a , for $1 \leq r, t \leq 3$.

Suppose that in step $n+1$ the guides are $i = i_{n+1}$ and $j = j_{n+1}$ and the idle string is $k = k_{n+1}$. As the string k is idle, the values of d_{kt} , $1 \leq t \leq 3$, corresponding to undercrossings of string k , do not change, so $d'_{kt} = d_{kt}$, for $1 \leq t \leq 3$.

Each time that string i crosses under string t , there are $2a$ new crossings under that string, $2 \lfloor a/2 \rfloor$ of which are formed by string i , or principal guide and $2 \lceil a/2 \rceil$ are formed by string j , or secondary guide, see Fig. 4 c.

Each time that string j crosses under string t , there are $2(a-1)$ new crossings under string t , $2 \lfloor (a-1)/2 \rfloor$ corresponding to crossings of string i and $2 \lceil (a-1)/2 \rceil$ corresponding to crossings of string j . See Fig. 4 c.

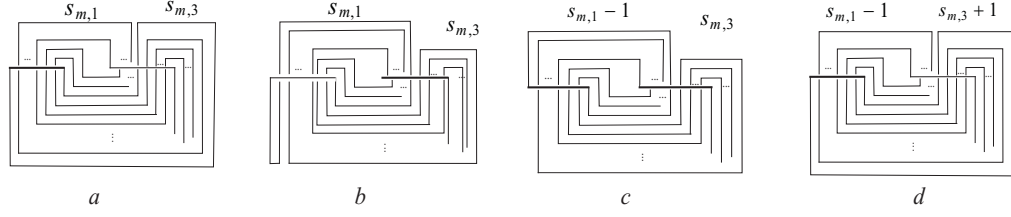


FIGURE 5. Final step. Figure a is for the case m even. Figures b, c and d are for m odd.

Besides these new crossings, when we arrive to the end of string i , we have a additional crossings, $\lfloor a/2 \rfloor$ corresponding to undercrossings of string i and $\lceil a/2 \rceil$ corresponding to undercrossings of string j . At the end of string j , there are $a-1$ additional crossings, $\lceil (a-1)/2 \rceil$ corresponding to string i and $\lfloor (a-1)/2 \rfloor$ corresponding to string j .

So at the end of step $n+1$ we have

$$\begin{aligned}
 d'_{kt} &= d_{kt}, \quad \text{for } 1 \leq t \leq 3, \\
 d'_{ik} &= \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor\right) d_{ik} + 2 \left\lceil \frac{a-1}{2} \right\rceil d_{jk}, \\
 d'_{ij} &= \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor\right) d_{ij} + 2 \left\lceil \frac{a-1}{2} \right\rceil d_{jj} + \left\lceil \frac{a-1}{2} \right\rceil, \\
 d'_{ii} &= \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor\right) d_{ii} + 2 \left\lceil \frac{a-1}{2} \right\rceil d_{ji} + \left\lfloor \frac{a}{2} \right\rfloor, \\
 d'_{jk} &= \left(1 + 2 \left\lceil \frac{a-1}{2} \right\rceil\right) d_{jk} + 2 \left\lfloor \frac{a}{2} \right\rfloor d_{ik}, \\
 d'_{jj} &= \left(1 + 2 \left\lceil \frac{a-1}{2} \right\rceil\right) d_{jj} + 2 \left\lfloor \frac{a}{2} \right\rfloor d_{ij} + \left\lfloor \frac{a-1}{2} \right\rfloor, \\
 d'_{ji} &= \left(1 + 2 \left\lceil \frac{a-1}{2} \right\rceil\right) d_{ji} + 2 \left\lfloor \frac{a}{2} \right\rfloor d_{ii} + \left\lceil \frac{a}{2} \right\rceil,
 \end{aligned}
 \tag{3.2}$$

where $\{i, j\}$ are the guides and k is the idle string.

The process just described can be written as a recurrence relation in terms of matrices, as we will describe in Theorem 2.

For the example $C[1, 2, 2]$ shown in Fig. 6, we have

$$D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} s_{1,1} \\ s_{1,2} \\ s_{1,3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} s_{2,1} \\ s_{2,2} \\ s_{2,3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

3.3. Final step. At the end of step m we have transformed all the tangles and we have produced a 2-bridge link. When m is even, we reach a reduced 2-bridge diagram but when m is odd, we reach a no reduced diagram with a kink formed by bridge α , that when simplified by a type I Reidemeister move produces a reduced 2-bridge diagram, see Fig. 5.

Now we interpret the meaning of the values $s_{m,1}$ and $s_{m,3}$ in the description of the final diagram.

When m is even, see Fig. 5 a., $s_{m,1}$ is the number of crossings under bridge α and $s_{m,3}$ is the number of crossings under string 3, so the position of the crossing under which the bridge α crosses bridge β is $s_{m,3}$, therefore we obtain the 2-bridge link of type

$$(s_{m,1} + 1) / s_{3,m}.$$

When m is odd, see Fig. 5 b., there is a kink formed by the bridge α and after the simplification by type I Reidemeister move, see Fig. 5 c., the number of crossings under bridge α decreased by one but the position of the crossing under which the bridge α crosses bridge β increases by one, see Fig. 5 d., and now it is $s_{m,3} + 1$, so we obtain the 2-bridge link

$$s_{m,1} / (s_{3,m} + 1).$$

In Fig. 6 e. to f. we show the final step in the transformation of $C[1, 2, 2]$. In Fig. 6 e. we change the tangle 2 and in Fig. 6 f. we move all the crossings following the guides. In Fig. 6 g. it is clear the kink that is simplified in Fig. 6 h.

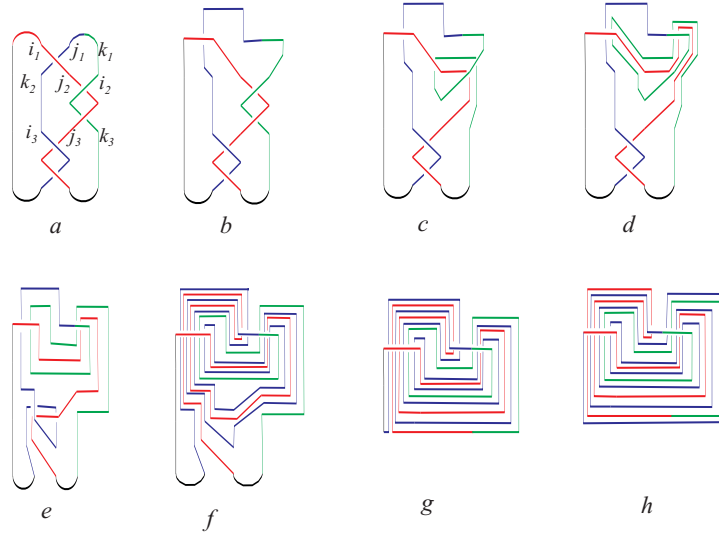


FIGURE 6. Transformation of $C[1, 2, 2]$ into the 2-bridge knot $7/5$.

For the example $C[1, 2, 2]$ we have

$$D_3 = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} s_{3,1} \\ s_{3,2} \\ s_{3,3} \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$$

and after the simplification we get 6 crossings under string 1 and 5 crossings under string 3, so we have the 2-bridge knot $7/5$.

3.4. Main result. In order to have the equivalence between the rational link $C[a_1, a_2, \dots, a_m]$ and the 2-bridge link p/q , where p/q is the rational given by $[a_1, a_2, \dots, a_m]$, we need to prove that the diagram obtained in the final step is in fact the diagram of the 2-bridge link p/q .

DEFINITION 3. Define $p_n = s_{n,1} + 1 - \mu_n$ and $q_n = s_{n,3} + \mu_n$, where μ_n was defined in (2.2).

So we need to prove that

$$p_m/q_m = p/q,$$

and we do so by proving the following theorem:

THEOREM 1 (Main Result). *Given the continued fraction $[a_1, a_2, \dots, a_m]$ that represents the rational p/q , the integers p_n and q_n defined above satisfy the recurrence relations*

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1}, \text{ for } 1 \leq n, \quad p_0 = 1, \quad p_1 = a_1, \\ q_{n+1} &= a_{n+1}q_n + q_{n-1}, \text{ for } 1 \leq n, \quad q_0 = 0, \quad q_1 = 1. \end{aligned}$$

Therefore p_n/q_n is the n -convergent of the continued fraction $[a_1, \dots, a_m]$ and so

$$p_m/q_m = p/q,$$

then the rational link $C[a_1, a_2, \dots, a_m]$ is equivalent to the 2-bridge link p/q .

The proof of this theorem is the content of the following sections.

4. Algorithmic description of the transformation process

In order to describe the transformation process the guides play a fundamental role. To each step n we assign a permutation $\sigma_n \in S_3$ that keep track of the guides.

DEFINITION 4. *We define a permutation of $\{1, 2, 3\}$ by*

$$(4.1) \quad \sigma_n(1) = i_n, \sigma_n(2) = j_n, \sigma_n(3) = k_n,$$

where i_n, j_n and k_n were defined in Definition 1, $1 \leq n \leq m$.

LEMMA 1. *The permutation σ_n satisfies the following recurrence relation*

$$(4.2) \quad \sigma_{n+1} = \begin{cases} \sigma_n(1\ 3), & \text{if } a_n \text{ is even,} \\ \sigma_n(1\ 3\ 2), & \text{if } a_n \text{ is odd,} \end{cases} \quad \text{for } n > 1 \text{ and } \sigma_1 = Id.$$

PROOF. Just consider the diagrams in Fig. 7.

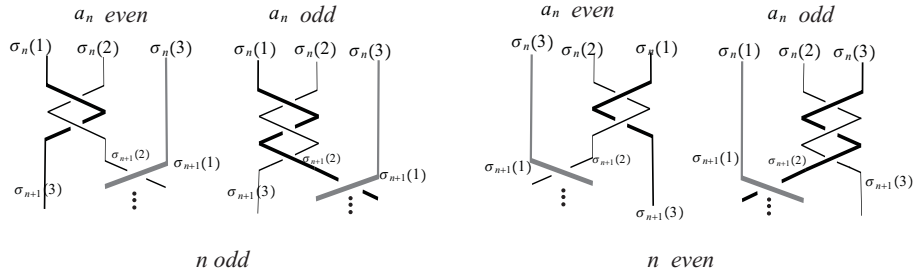


FIGURE 7. Changes from permutation σ_n to permutation σ_{n+1} .

□

COROLLARY 1. *For $n > 1$, $i_{n+1} = k_n$ and if a_n is even then $j_{n+1} = j_n$ and $k_{n+1} = i_n$ and if a_n is odd then $j_{n+1} = i_n$ and $k_{n+1} = j_n$.*

DEFINITION 5. For each step n , with guides $i_n = i, j_n = j$ and tangle a_n , we define the matrix $M_n = M(a_n, i, j) = [m_{rt}] \in M_{3 \times 3}(\mathbb{N})$ by

$$m_{ii} = \lfloor a_n/2 \rfloor, \quad m_{ij} = \lceil (a_n - 1)/2 \rceil, \quad m_{ji} = \lceil a_n/2 \rceil, \quad m_{jj} = \lfloor (a_n - 1)/2 \rfloor$$

and zero elsewhere.

The following lemma is clear and give us an alternative way to compute M_n .

LEMMA 2. The matrix M_n can be described as

$$M_n = P_{\sigma_n} M(a_n) P_{\sigma_n}^{-1}.$$

where $M(a_n) = \begin{bmatrix} \lfloor a_n/2 \rfloor & \lceil (a_n - 1)/2 \rceil & 0 \\ \lceil a_n/2 \rceil & \lfloor (a_n - 1)/2 \rfloor & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and P_{σ_n} is the permutation matrix of σ_n .

THEOREM 2. The associated matrix D_n of the transformation process of changing the diagram C of the rational link into a 2-bridge diagram satisfies the recurrence relation

$$(4.3) \quad D_{n+1} = 2M_{n+1}D_n + D_n + M_{n+1}, \quad 0 \leq n \leq m$$

where M_{n+1} is the changing matrix associated to a_{n+1} and to the permutation σ_{n+1} , $D_0 = [0]$ and $\sigma_1 = Id$.

PROOF. This is just a new way to write the relations given in (3.2). \square

The recurrence described in Theorem 2 is very easy to implement in a computer, in our case we use *Mathematica*. The following is an example of the calculation to transform the rational diagram $C[2, 3, 1, 2, 3]$.

EXAMPLE 1. Change the rational $C[2, 5, 4, 1]$ into the 2-bridge $57/26$.

Step n	a_n	i_n	j_n	M_n	D_n	p_n	q_n
1	2	1	2	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	2	1
2	5	3	2	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 5 & 2 & 3 \\ 4 & 2 & 2 \end{bmatrix}$	11	5
3	4	1	3	$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 23 & 13 & 10 \\ 5 & 2 & 3 \\ 18 & 10 & 7 \end{bmatrix}$	46	21
4	1	2	3	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 23 & 13 & 10 \\ 5 & 2 & 3 \\ 28 & 15 & 13 \end{bmatrix}$	57	26

Table 1

4.1. Properties of matrix D_n . As Example 1 shows, there are some interesting patterns in the entries of matrix D_n . For instance, the column 1 is "almost" the sum of columns 2 and 3. In this section we will find relations among the entries of matrix D_n and among the entries of D_n and D_{n+1} . The results of this section will be used in the proof of the main result. All the results depend on the guides i_{n+1}, j_{n+1} and the parity of n and a_n .

PROPOSITION 1. *Given the rational link $C[a_1, a_2, \dots, a_m]$, the sequence of integers $s_{n,t}$ satisfies the recurrence relation*

$$s_{n+1,t} = a_{n+1}s_{n,t} + a_{n+1}(d_{it} + d_{jt} - d_{kt} + \delta_{it} + \delta_{jt}) + (d_{it} - d_{jt} + d_{kt} - \delta_{jt}),$$

$1 \leq n \leq m, 1 \leq t \leq 3$, where $i = i_{n+1}, j = j_{n+1}, k = k_{n+1}$ and δ_{rt} is the Kronecker delta.

PROOF. Let $D_n = [d_{rt}]$ and $D' = D_{n+1} = [d'_{rt}]$ the data matrices of steps n and $n+1$ in the transformation process; besides let $i = i_{n+1}, j = j_{n+1}$ be the guides, $k = k_{n+1}$ be the idle string in step $n+1$ and $a = a_{n+1}$. By (3.2) we have,

$$(4.4) \quad \begin{aligned} s_{n+1,k} &= d'_{ik} + d'_{jk} + d'_{kk} = \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor + 2 \left\lceil \frac{a}{2} \right\rceil\right) d_{ik} + \\ &\quad \left(1 + 2 \left\lfloor \frac{a-1}{2} \right\rfloor + 2 \left\lceil \frac{a-1}{2} \right\rceil\right) d_{jk} + d_{kk}, \\ s_{n+1,i} &= d'_{ii} + d'_{ji} + d'_{ki} = \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor + 2 \left\lceil \frac{a}{2} \right\rceil\right) d_{ii} + \\ &\quad \left(1 + 2 \left\lfloor \frac{a-1}{2} \right\rfloor + 2 \left\lceil \frac{a-1}{2} \right\rceil\right) d_{ji} + d_{ki} + \left\lfloor \frac{a}{2} \right\rfloor + \left\lceil \frac{a}{2} \right\rceil, \\ s_{n+1,j} &= d'_{ij} + d'_{jj} + d'_{kj} = \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor + 2 \left\lceil \frac{a}{2} \right\rceil\right) d_{ij} + \\ &\quad \left(1 + 2 \left\lfloor \frac{a-1}{2} \right\rfloor + 2 \left\lceil \frac{a-1}{2} \right\rceil\right) d_{jj} + d_{kj} + \left\lfloor \frac{a-1}{2} \right\rfloor + \left\lceil \frac{a-1}{2} \right\rceil, \end{aligned}$$

but as $1 + 2 \left\lfloor \frac{a}{2} \right\rfloor + 2 \left\lceil \frac{a}{2} \right\rceil = 1 + 2a$ and $\left\lfloor \frac{a}{2} \right\rfloor + \left\lceil \frac{a}{2} \right\rceil = a$ we have

$$(4.5) \quad s_{n+1,t} = \begin{cases} (1 + 2a) d_{ik} + (2a - 1) d_{jk} + d_{kk}, & \text{if } t = k, \\ (1 + 2a) d_{ii} + (2a - 1) d_{ji} + d_{ki} + a, & \text{if } t = i, \\ (1 + 2a) d_{ij} + (2a - 1) d_{jj} + d_{kj} + a - 1, & \text{if } t = j, \end{cases}$$

therefore

$$\begin{aligned} s_{n+1,t} &= (1 + 2a) d_{it} + (2a - 1) d_{jt} + d_{kt} + a \delta_{it} + (a - 1) \delta_{jt} \\ &= a(d_{it} + d_{jt} + d_{kt}) + a(d_{it} + d_{jt} - d_{kt} + \delta_{it} + \delta_{jt}) + (d_{it} - d_{jt} + d_{kt} - \delta_{jt}) \\ &= a s_{n,t} + a(d_{it} + d_{jt} - d_{kt} + \delta_{it} + \delta_{jt}) + (d_{it} - d_{jt} + d_{kt} - \delta_{jt}). \end{aligned}$$

□

Now we need to find expressions for $(d_{it} + d_{jt} - d_{kt} + \delta_{it} + \delta_{jt})$ and $(d_{it} - d_{jt} + d_{kt} - \delta_{jt})$. The following lemmas will take care of finding these expressions that will be used in proving the main result.

LEMMA 3. *If $i = i_{n+1}, j = j_{n+1}, k = k_{n+1}$ and $a = a_{n+1}$, then*

$$(4.6) \quad d'_{it} - d'_{jt} = (-1)^{\mu_a} (d_{it} + d_{jt} + \mu_a \delta_{it} + (1 - \mu_a) \delta_{jt}), \quad \text{for } 1 \leq t \leq 3,$$

where μ_a is given by (2.2).

PROOF. Suppose $i = i_{n+1}, j = j_{n+1}, k = k_{n+1}$ and $a = a_{n+1}$. Using the relations given in (3.2) we get

$$d'_{ik} - d'_{jk} = \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor - 2 \left\lceil \frac{a}{2} \right\rceil\right) d_{ik} + \left(-1 + 2 \left\lfloor \frac{a-1}{2} \right\rfloor - 2 \left\lceil \frac{a-1}{2} \right\rceil\right) d_{jk},$$

$$\begin{aligned}
d'_{ii} - d'_{ji} &= \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor - 2 \left\lceil \frac{a}{2} \right\rceil\right) d_{ii} + \left(-1 + 2 \left\lceil \frac{a-1}{2} \right\rceil - 2 \left\lfloor \frac{a-1}{2} \right\rfloor\right) d_{ji} + \\
&\quad \left(\left\lfloor \frac{a}{2} \right\rfloor - \left\lceil \frac{a}{2} \right\rceil\right), \\
d'_{ij} - d'_{jj} &= \left(1 + 2 \left\lfloor \frac{a}{2} \right\rfloor - 2 \left\lceil \frac{a}{2} \right\rceil\right) d_{ij} + \left(-1 + 2 \left\lceil \frac{a-1}{2} \right\rceil - 2 \left\lfloor \frac{a-1}{2} \right\rfloor\right) d_{jj} + \\
&\quad \left(\left\lceil \frac{a-1}{2} \right\rceil - \left\lfloor \frac{a-1}{2} \right\rfloor\right),
\end{aligned}$$

but $\left\lfloor \frac{a}{2} \right\rfloor - \left\lceil \frac{a}{2} \right\rceil = -\mu_a = (-1)^{\mu_a} \mu_a$, $\left\lceil \frac{a-1}{2} \right\rceil - \left\lfloor \frac{a-1}{2} \right\rfloor = 1 - \mu_a = (-1)^{\mu_a} (1 - \mu_a)$ and $1 + 2 \left\lfloor \frac{a}{2} \right\rfloor - 2 \left\lceil \frac{a}{2} \right\rceil = -1 + 2 \left\lceil \frac{a-1}{2} \right\rceil - 2 \left\lfloor \frac{a-1}{2} \right\rfloor = (-1)^{\mu_a}$, therefore

$$\begin{aligned}
(4.7) \quad d'_{ik} - d'_{jk} &= (-1)^{\mu_a} (d_{ik} + d_{jk}) \\
d'_{ii} - d'_{ji} &= (-1)^{\mu_a} (d_{ii} + d_{ji}) - \mu_a \\
d'_{ij} - d'_{jj} &= (-1)^{\mu_a} (d_{ij} + d_{jj}) + 1 - \mu_a.
\end{aligned}$$

Also, as

$$(-1)^{\mu_a} (\mu_a \delta_{it} + (1 - \mu_a) \delta_{jt}) = \begin{cases} 0, & \text{if } t = k \\ -\mu_a, & \text{if } t = i \\ 1 - \mu_a, & \text{if } t = j \end{cases}$$

we get (4.6). \square

The following corollary will be used in the proof of Theorem 3. Later, we will consider only the values of the sum of only two of the columns of D_n , because the other one will give redundant information.

COROLLARY 2. *If $i = i_{n+1}$, $j = j_{n+1}$, $k = k_{n+1}$ and $a = a_{n+1}$ then*

$$(4.8) \quad s_{n,t} = (-1)^{\mu_a} (d'_{it} - d'_{jt}) + d'_{kt} - \mu_a \delta_{it} + (\mu_a - 1) \delta_{jt}, \quad \text{for } 1 \leq t \leq 3,$$

where $s_{n,t} = d_{1t} + d_{2t} + d_{3t}$.

PROOF. It is enough to recall that $k = k_{n+1}$ is the idle string, so $d'_{kt} = d_{kt}$, therefore from Lemma 3 we get,

$$(-1)^{\mu_a} (d'_{it} - d'_{jt}) + d'_{kt} = d_{it} + d_{jt} + d_{kt} + \mu_a \delta_{it} + (1 - \mu_a) \delta_{jt}, \quad \text{for } 1 \leq t \leq 3.$$

\square

LEMMA 4. *If $i = i_{n+1}$, $j = j_{n+1}$ and $k = k_{n+1}$ then*

$$(4.9) \quad s_{n-1,t} = d_{it} - d_{jt} + d_{kt} - \delta_{jt}$$

PROOF. We apply Corollary 2 to the step n .

For a_n even, $\mu_{a_n} = 0$ and by Corollary 1, $i_n = k_{n+1} = k$, $j_n = j_{n+1} = j$ and $k_n = i_{n+1} = i$, so we get by (4.8)

$$s_{n-1,t} = (-1)^{\mu_{a_n}} (d_{kt} - d_{jt}) + d_{it} - \mu_{a_n} \delta_{kt} + (\mu_{a_n} - 1) \delta_{jt} = d_{kt} - d_{jt} + d_{it} - \delta_{jt}.$$

For a_n odd, $\mu_{a_n} = 1$ and by Corollary 1 $i_n = j_{n+1} = j$, $j_n = k_{n+1} = k$ and $k_n = i_{n+1} = i$, so by (4.8)

$$s_{n-1,t} = (-1)^{\mu_{a_n}} (d_{jt} - d_{kt}) + d_{it} - \mu_{a_n} \delta_{jt} + (\mu_{a_n} - 1) \delta_{kt} = -d_{jt} + d_{kt} + d_{it} - \delta_{jt},$$

therefore, in both cases we get (4.9). \square

This lemma tell us that the values of $s_{n-1,t}$ can be computed using the matrix D_n .

LEMMA 5. *If $i = i_{n+1}, j = j_{n+1}, k = k_{n+1}$ then*

$$(4.10) \quad d_{kt} = d_{it} + d_{jt} - \delta_{kt} + \delta_{3t} + (-1)^{\delta_{3t}} \mu_n, \quad 1 \leq t \leq 3,$$

where $\mu_n = 0$ if n is even and $\mu_n = 1$ if n is odd.

PROOF. The proof is by induction on n . Notice that condition (4.10) corresponds to the following three conditions:

$$\begin{aligned} d_{k1} &= d_{i1} + d_{j1} - \delta_{k1} + \mu_n, \\ d_{k2} &= d_{i2} + d_{j2} - \delta_{k2} + \mu_n \\ d_{k3} &= d_{i3} + d_{j3} - \delta_{k3} + 1 - \mu_n. \end{aligned}$$

For $n = 0$, $D_0 = [0]$, and the guides for step one are $i = i_1 = 1, j = j_1 = 2$ and the idle string is $k = k_1 = 3$. Then

$$\begin{aligned} d_{31} &= 0 = d_{11} + d_{21} - \delta_{31} + \mu_0, \\ d_{32} &= 0 = d_{12} + d_{22} - \delta_{32} + \mu_0, \\ d_{33} &= 0 = d_{13} + d_{23} - \delta_{33} + 1 - \mu_0. \end{aligned}$$

For $n = 1$,

$$(4.11) \quad D_1 = \begin{bmatrix} \left\lfloor \frac{a_1}{2} \right\rfloor & \left\lfloor \frac{a_1 - 1}{2} \right\rfloor & 0 \\ \left\lceil \frac{a_1}{2} \right\rceil & \left\lceil \frac{a_1 - 1}{2} \right\rceil & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If a_1 is even, by Corollary 1, $i = i_2 = 3, j = j_2 = 2$ and $k = k_2 = 1$, then

$$\begin{aligned} d_{11} &= \left\lfloor \frac{a_1}{2} \right\rfloor = \left\lceil \frac{a_1}{2} \right\rceil = d_{31} + d_{21} - \delta_{11} + \mu_1, \\ d_{12} &= \left\lceil \frac{a_1 - 1}{2} \right\rceil = \left\lfloor \frac{a_1 - 1}{2} \right\rfloor + 1 = d_{32} + d_{22} - \delta_{12} + \mu_1, \\ d_{13} &= 0 = d_{33} + d_{23} - \delta_{13} + 1 - \mu_1. \end{aligned}$$

If a_1 is odd, by Corollary 1, $i = i_2 = 3, j = j_2 = 1$ and $k = k_2 = 2$ therefore

$$\begin{aligned} d_{21} &= \left\lceil \frac{a_1}{2} \right\rceil = \left\lfloor \frac{a_1}{2} \right\rfloor + 1 = d_{31} + d_{11} + 1 = d_{31} + d_{11} - \delta_{21} + \mu_1, \\ d_{22} &= \left\lfloor \frac{a_1 - 1}{2} \right\rfloor = \left\lceil \frac{a_1 - 1}{2} \right\rceil = d_{32} + d_{12} - \delta_{22} + \mu_1, \\ d_{23} &= 0 = d_{33} + d_{23} - \delta_{23} + 1 - \mu_1. \end{aligned}$$

Suppose that $i = i_{n+1}, j = j_{n+1}, k = k_{n+1}$ and that relations (4.10) hold for step n . To prove that the results are valid for step $n + 1$ we need to find the $n + 2$ step guides, that depend on whether a_{n+1} is even or odd.

If a_{n+1} is even by Corollary 1, $i_{n+2} = k_{n+1} = k, j_{n+2} = j_{n+1} = j$ and $k_{n+2} = i_{n+1} = i$ and we need to prove that

$$d'_{it} = d'_{jt} + d'_{kt} - \delta_{i1} + \delta_{3t} + (-1)^{\delta_{3t}} \mu_{n+1}, \quad \text{for } 1 \leq t \leq 3.$$

By Lemma 3 we have

$$d'_{it} - d'_{jt} = d_{it} + d_{jt} + \delta_{jt}$$

As k is the idle string in the $n + 1$ step, we have that $d'_{kt} = d_{kt}$ and for (4.10) we get

$$(4.12) \quad d_{it} + d_{jt} = d'_{kt} + \delta_{kt} - \delta_{3t} - (-1)^{\delta_{3t}} \mu_n,$$

so

$$d'_{it} = d'_{jt} + (d_{it} + d_{jt}) + \delta_{jt} = d'_{jt} + \left(d'_{kt} + \delta_{kt} - \delta_{3t} - (-1)^{\delta_{3t}} \mu_n \right) + \delta_{jt},$$

but $\delta_{kt} + \delta_{jt} = 1 - \delta_{it}$ and

$$(4.13) \quad -\delta_{3t} - (-1)^{\delta_{3t}} \mu_n = -\delta_{3t} - (-1)^{\delta_{3t}} (1 - \mu_{n+1}) = \delta_{3t} - 1 + (-1)^{\delta_{3t}} \mu_{n+1},$$

so

$$d'_{it} = d'_{jt} + d'_{kt} + 1 - \delta_{it} - \delta_{3t} - (-1)^{\delta_{3t}} \mu_n = d'_{jt} + d'_{kt} - \delta_{it} + \delta_{3t} + (-1)^{\delta_{3t}} \mu_{n+1}.$$

If $a = a_{n+1}$ is odd, by Corollary 1 $i_{n+2} = k_{n+1} = k$, $j_{n+2} = i_{n+1} = i$ and $k_{n+2} = j_{n+1} = j$.

By Lemma 3 we have

$$d'_{it} - d'_{jt} = -d_{it} - d_{jt} - \delta_{it}$$

thus, by (4.12) and (4.13) we have

$$\begin{aligned} d'_{jt} &= d'_{it} + (d_{it} + d_{jt}) + \delta_{it} = d'_{it} + (d'_{kt} + \delta_{kt} - \delta_{3t} - (-1)^{\delta_{3t}} \mu_n) + \delta_{it} \\ &= d'_{it} + d'_{kt} + \delta_{kt} + \delta_{it} - 1 + \delta_{3t} + (-1)^{\delta_{3t}} \mu_{n+1} \\ &= d'_{it} + d'_{kt} + \delta_{jt} + \delta_{3t} + (-1)^{\delta_{3t}} \mu_{n+1}. \end{aligned}$$

□

COROLLARY 3. *If $k = k_{n+1}$ then*

$$s_{n,t} = 2d_{kt} + \delta_{kt} - \delta_{3t} - (-1)^{\delta_{3t}} \mu_n, \quad 1 \leq t \leq 3.$$

5. Proof of Main Result

Now we use the results of the previous section to prove the main result:

THEOREM 3 (Main Result). *Given the continued fraction $[a_1, a_2, \dots, a_m]$ that represents the rational p/q , the integers p_n and q_n defined in Definition 3 satisfy the recurrence relations*

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \text{ for } 1 \leq n, \quad p_0 = 1, \quad p_1 = a_1,$$

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \text{ for } 1 \leq n, \quad q_0 = 0, \quad q_1 = 1.$$

Therefore p_n/q_n is the n -convergent of the continued fraction $[a_1, \dots, a_m]$ and so

$$p_m/q_m = p/q,$$

therefore the rational link $C[a_1, a_2, \dots, a_m]$ is equivalent to the 2-bridge link p/q .

PROOF. Let $[a_1, \dots, a_m]$ be a continued fraction. For $n = 0$, $D_0 = [0]$, so $p_0 = 1$ and $q_0 = 0$. From (4.11), $s_{1,1} = a_1$ and $s_{3,1} = 0$, so $p_1 = a_1$ and $q_1 = 1$.

By Proposition 1

$$s_{n+1,t} = a_{n+1}s_{n,t} + a_{n+1}(d_{it} + d_{jt} - d_{kt} + \delta_{it} + \delta_{jt}) + (d_{it} - d_{jt} + d_{kt} - \delta_{jt})$$

If $i = i_{n+1}$, $j = j_{n+1}$, $k = k_{n+1}$ by Lemma 5

$$0 = d_{it} + d_{jt} - d_{kt} - \delta_{kt} + \delta_{3t} + (-1)^{\delta_{3t}} \mu_n,$$

but $\delta_{it} + \delta_{jt} = 1 - \delta_{kt}$, so

$$d_{it} + d_{jt} - d_{kt} - \delta_{kt} = d_{it} + d_{jt} - d_{kt} + \delta_{it} + \delta_{jt} - 1 = -\delta_{3t} - (-1)^{\delta_{3t}} \mu_n$$

and by Lemma 4

$$s_{n-1,t} = d_{it} - d_{jt} + d_{kt} - \delta_{jt}$$

so

$$(5.1) \quad s_{n+1,t} = a_{n+1}s_{n,t} + a_{n+1} \left(1 - \delta_{3t} - (-1)^{\delta_{3t}} \mu_n \right) + s_{n-1,t}$$

then, for $t = 1$, $s_{n,1} = p_n - 1 + \mu_n$

$$p_{n+1} - 1 + \mu_{n+1} = a_{n+1} (p_n - 1 + \mu_n) + a_{n+1} (1 - \mu_n) + p_{n-1} - 1 + \mu_{n-1}$$

and $\mu_{n+1} = \mu_{n-1}$ so

$$p_{n+1} = a_{n+1}p_n + p_{n-1}$$

For $t = 3$, $s_{n,3} = q_n - \mu_n$, so (5.1) becomes

$$q_{n+1} - \mu_{n+1} = a_{n+1} (q_n - \mu_n) + a_{n+1} (1 - 1 + \mu_n) + q_{n-1} - \mu_{n-1}$$

therefore

$$q_{n+1} = a_{n+1}q_n + q_{n-1}.$$

□

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